ON p-ABSOLUTELY SUMMING CONSTANTS OF BANACH SPACES*

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ABSTRACT

Given $1 \leq p < \infty$ and a real Banach space X, we define the p-absolutely summing constant $\mu_p(X)$ as $\inf\{\sup[\sum_{i=1}^m |x^*(x_i)|^p / \sum_{i=1}^m \|x_i\|^p]^{1/p}\}$, where the supremum ranges over $\{x^* \in X^*;\ \|x^*\| \leq 1\}$ and the infimum is taken over all sets $\{x_1, x_2, \ldots, x_m\} \subset X$ such that $\sum_{i=1}^m \|x_i\| > 0$. It follows immediately from [2] that $\mu_p(X) > 0$ if and only if X is finite dimensional. In this paper we find the exact values of $\mu_p(X)$ for various spaces, and obtain some asymptotic estimates of $\mu_p(X)$ for general finite dimensional Banach spaces.

1. **Preliminaries and definitions.** The results obtained here are in part related to those in [3]. We recall briefly some basic definitions: the projection constant of a Banach space X is defined as $\lambda(X) = \inf\{\lambda > 0$; from every Banach space $Y \supset X$, there is a projection onto X with norm $\leqq \lambda\}$. The Macphail constant is defined as $\mu(X) = \inf\{\sup_J \| \sum_{j \in J} x_j \| / \sum_{j=1}^m \| x_j \|\}$, where J ranges on the subsets of $\{1, 2, \cdots, m\}$, and the infimum is taken over all finite sets $\{x_1, x_2, \cdots, x_m\} \subset X$ such that $\sum_{j=1}^m \| x_j \| > 0$. The distance $d(X, Y)$ between isomorphic Banach spaces X and Y is defined as $\inf\{\| T \| \, \| T^{-1} \|$; T is an isomorphism of X onto $Y\}$. l_n^p $(1 \leqq p < \infty)$ denotes the space of real n-tuples $x = (x_1, x_2, \cdots, x_n)$ with the norm $\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$; l_n^∞ denotes the same space with the norm $\| x \|_\infty = \max_i |x_i|$. All the asymptotic values of $\mu(l_n^p)$, $\lambda(l_n^p)$, $d(l_n^p, l_n^q)$ $(1 \leqq p,\ q \leqq \infty)$ are now known (see [3] for a short summary).

2. **Formulation of results.** We state here the main results which are to be proved.

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THEOREM 1. *Given a Banach space X, let K^* be the ω^* closure of the set of all the extremal points of the unit ball of X^*. For any $1 \leqq p < \infty$, there is a probability measure* (i.e. *a regular non-negative Borel measure with total mass* 1) *ν over K^*, such that*

$$(1) \qquad \mu_p(X) = \inf_{\|x\|=1} \left(\int_{K^*} |x^*(x)|^p d\nu(x^*) \right)^{1/p}.$$

Moreover, for every probability measure θ over K^* and every $1 \leqq p < \infty$, the following inequality holds

$$\mu_p(X) \geqq \inf_{\|x\|=1} \left(\int_{K^*} |x^*(x)|^p d\theta(x^*) \right)^{1/p}.$$

THEOREM 2. *If $1 \leqq p < \infty$, then*

$$(2) \qquad \mu_p(l_n^\infty) = n^{-1/p},$$

$$(3) \qquad \mu_p(l_n^2) = [\Gamma(n/2)\Gamma(p/2+1/2)/\Gamma(1/2)\Gamma(n/2+p/2)]^{1/p},$$

$$(4) \qquad \mu_p(l_n^1) = \left[2^{-n} n^{-p} \sum_{k=0}^{n} \binom{n}{k} |n-2k|^p \right]^{1/p},$$

$$(5) \qquad \mu_p(G_{2n}) = \left[(2n)^{-1} \sum_{i=0}^{2n-1} |\cos(i\pi/n)|^p \right]^{1/p},$$

where G_{2n} is the space whose unit ball is the affine-regular $2n$-sided polygon in the Minkowsky plane.

THEOREM 3. *Let X be an n-dimensional real Banach space, then*

$$(6) \qquad \mu_p(X) \leqq n^{-1/4}, \qquad \text{if } 1 \leqq p \leqq 2,$$

$$(7) \qquad \mu_p(X) \leqq c_p n^{-1/(2+p)}, \qquad \text{if } 2 \leqq p < \infty,$$

$$(8) \qquad \mu_p(X)\mu_p(X^*) \leqq c_p n^{-2/3}, \qquad \text{if } 1 \leqq p \leqq 2,$$

$$(9) \qquad \mu_p(X)\mu_p(X^*) \leqq c_p n^{-4/(4+p)}, \qquad \text{if } 2 \leqq p \leqq 4,$$

$$(10) \qquad \mu_p(X)\mu_p(X^*) \leqq c_p n^{-1/2}, \qquad \text{if } 4 \leqq p < \infty,$$

(c_p denotes a constant depending only on p).

THEOREM 4. *Let X be an n-dimensional real Banach space, and K_G the universal Grothendieck constant* ($\pi/2 \leqq K_G \leqq \sinh(\pi/2)$). *Then*

(11) $K_G\mu_1(X)d(X,l_n^2)d(X,l_n^1) \geqq 1.$

(12) $K_G\mu_2(X)d(X,l_n^p)\lambda(X) \geqq 1,$ *for every* $1 \leqq p \leqq 2.$

We say that two functions f, g defined on the integers are asymptotically equivalent and write $f(n) \sim g(n)$ if $\sup_n(f(n)/g(n)) < \infty$ and $\inf_n(f(n)/g(n)) > 0$.

THEOREM 5. *For fixed p and r,*

$$\mu_p(l_n^r) \sim \begin{cases} n^{-1/2} & ; \text{ if } 1 < r \leqq 2, 1 \leqq p < \infty \\ n^{-1/r} & ; \text{ if } 2 \leqq r \leqq p < \infty \\ n^{-1/p} & ; \text{ if } r/(r-1) \leqq p \leqq r < \infty \\ n^{-1+1/r} & ; \text{ if } 1 \leqq p \leqq r/(r-1) \leqq r < \infty \end{cases}$$

REMARKS. Theorem 1 is a consequence of a result due to Pietsch [7] who introduced the notion of p-absolutely summing operators, a variant of his result may also be found in [6, Proposition 3.1]. The proofs of Theorems 2, 3 are based on Theorem 1, and Theorem 3 generalizes a consequence of [2, Theorem 4] which states that $\mu_1(X) \leqq 2n^{-1/4}$ if X is n-dimensional. Theorem 4 is an application of Theorems 4.1 and 4.3 of [6]; (11) was essentially proved in [3]. Theorem 5 furnishes the asymptotic behaviour of $\mu_p(l_n^r)$ for all fixed values of $1 \leqq p < \infty$ and $1 < r < \infty$.

3. **Proof of Theorem 1 and its consequences.** Let $C(K^*)$ be the space of real continuous functions on the set K^* which is compact in its ω^* topology. For every $x \in X$, let $\phi_x \in C(K^*)$ be defined by $\phi_x(x^*) = |x^*(x)|^p$. Assuming that $\mu_p(X) > 0$, let M be the closed convex hull of the set $\{\mu_p^{-p}(X)\phi_x; \|x\| = 1\} \subset C(K^*)$, and let $N = \{f \in C(K^*); \sup_{x^* \in K^*} f(x^*) < 1\}$. It is easily verified that M and N are disjoint convex sets, hence there is a functional in $C^*(K^*)$, i.e. a regular Borel measure v' on K^*, such that $\int f dv' \geqq 1 > \int g dv'$ for every $f \in M$, $g \in N$. Since N contains the negative functions and the open unit ball of $C(K^*)$, it follows that v' is non-negative and that $v' = av$ where $0 < a \leqq 1$, and v is a probability measure. Therefore, $\int f dv \geqq 1$ for every $f \in M$. The proof of the second part of Theorem 1 is trivial and yields the equality in Equation (1).

COROLLARY 1. *If* $1 \leqq p \leqq q < \infty$, *then*

(13) $$\mu_p^q(X) \leqq \mu_q^q(X) \leqq \mu_p^p(X).$$

Proof. Using Hölder's inequality in (1), we get for every $x \in X$,

$$\mu_p(X)\|x\| \leqq \left(\int_{K^*}|x^*(x)|^p d\nu(x^*)\right)^{1/p} \leqq \left(\int_{K^*}|x^*(x)|^q d\nu(x^*)\right)^{1/q}$$

and by the second part of Theorem 1, $\mu_p(X) \leqq \mu_q(X)$.

Again, use of (1) yields

$$\mu_q^q(X)\|x\|^q \leqq \int_{K^*}|x^*(x)|^q d\nu(x^*) \leqq \|x\|^{q-p}\int_{K^*}|x^*(x)|^p d\nu(x^*), \text{ hence}$$

$$\mu_q^{q/p}(X)\|x\| \leqq \left(\int_{K^*}|x^*(x)|^p d\nu(x^*)\right)^{1/p}, \text{ therefore } \mu_q^{q/p}(X) \leqq \mu_p(X).$$

COROLLARY 2. *If X is n-dimensional, then*

$$\mu_2(X)d(X, l_n^2) \leqq 1. \tag{14}$$

Proof. Assuming that $\mu_2(X) > 0$, the inequality

$$\mu_2(X)\|x\| \leqq \left(\int_{K^*}|x^*(x)|^2 d\nu(x^*)\right)^{1/2} \leqq \|x\|,$$

implies that X is isomorphic to an n-dimensional subspace of the Hilbert space $L_2(K^*, \nu)$, and since such a subspace is isometric to l_n^2, (13) follows immediately.

4. **Proof of Theorem 2.** We first establish the following Lemma.

LEMMA. *Let X be an n-dimensional real Banach space. There is a probability measure ν^* which satisfies* (1), *and which is invariant under all isometries of X.*

Proof. Let ν be any probability measure which satisfies (1), and let $0X$ be the compact topological group of all isometries of X onto itself. If $T \in 0X$, then $T^*K^* = K^*$, and denoting by ν_T the measure defined by $d\nu_T(x^*) = d\nu(T^*x^*)$, we have for every $x \in X$

$$\mu_p(X)\|Tx\| = \mu_p(X)\|x\| \leqq \left(\int_{K^*}|x^*(x)|^p d\nu(x^*)\right)^{1/p}$$

$$= \left(\int_{K^*}|T^*x^*(x)|^p d\nu(T^*x^*)\right)^{1/p} = \left(\int_{K^*}|x^*(Tx)|^p d\nu_T(x^*)\right)^{1/p},$$

and thus we get

$$\mu_p(X)\|x\| \leqq \left(\int_{K^*}|x^*(x)|^p d\nu_T(x^*)\right)^{1/p}, \text{ for every } x \in X, T \in 0X.$$

Let dm denote the unique normalized non-negative Haar measure defined for the compact topological group $0X$, and define the measure v^* by $v^*(A) = \int_{0X} v_T(A)\,dm(T)$, where A is any Borel subset of K^*. Obviously v^* is a probability measure which satisfies (1), and for any $S \in 0X$ and any Borel subset $A \subseteq K^*$, we have

$$\text{(15)} \qquad v^*(S^*A) = \int_{0X} v_T(S^*A)dm(T) = \int_{0X} v_{ST}(A)dm(T)$$

$$= \int_{0X} v_{ST}(A)dm(ST) = v^*(A).$$

Proof of Theorem 2. Let e_i and e_i^* $(1 \leqq i \leqq n)$ denote the usual basis of l_n^∞ and $(l_n^\infty)^* = l_n^1$ respectively. Obviously K^* is the set $\{e_1^*, -e_1^*, \cdots, e_n^*, -e_n^*\}$. By (15), $v^*(\{e_i^*\}) = v^*(\{-e_i^*\}) = 1/2n$ for every $1 \leqq i \leqq n$, and by Theorem 1

$$\mu_p^p(l_n^\infty) = \inf\left(\int_{K^*} |x^*(x)|^p dv^*(x^*);\ \|x\|_\infty = 1\right)$$

$$= \inf\left(n^{-1} \sum_{i=1}^{n} |\xi_i|^p;\ \max_i |\xi_i| = 1\right) = n^{-1},$$

this establishes (2).

In the case of l_n^2, $K^* = S_{n-1} = \{x; \|x\|_2 = 1\}$, and by (15), v^* is the usual normalized $(n-1)$-dimensional measure defined on S_{n-1}. Denoting this measure by dm_{n-1}, we have by Theorem 1

$$\mu_p^p(l_n^2) = \inf\left\{\int_{S_{n-1}} |(x,y)|^p dm_{n-1}(y);\ \|x\|_2 = 1\right\}.$$

This integral is independent of the choice of $x \in S_{n-1}$, and taking $x = (1,0,\cdots,0)$ let $y = (y_1,\cdots,y_n) \in S_{n-1}$ be defined by spherical coordinates

$y_1 = \sin\theta_1$, $y_2 = \cos\theta_1 \sin\theta_2, \cdots, y_{n-1} = \cos\theta_1 \cdots \cos\theta_{n-2}\sin\theta_{n-1}$,

$y_n = \cos\theta_1 \cdots \cos\theta_{n-1}$, $-\pi/2 \leqq \theta_i \leqq \pi/2 \quad (1 \leqq i \leqq n-2)$,

$-\pi \leqq \theta_{n-1} \leqq \pi$, and $dm_{n-1}(y) = \sigma_{n-1}^{-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \cdots \cos\theta_{n-2}\, d\theta_1 \cdots d\theta_{n-1}$,

where σ_{n-1} is the usual $(n-1)$-dimensional measure of S_{n-1}. Now,

$$\mu_p^p(l_n^2) = \int_{S_{n-1}} |(x,y)|^p dm_{n-1}(y) = 2\sigma_{n-1}^{-1}\sigma_{n-2}\int_0^{\pi/2} \sin^p\theta_1 \cos^{n-2}\theta_1 d\theta_1$$

$$= \text{(substituting, } t = \sin^2\theta_1)\, \sigma_{n-1}^{-1}\sigma_{n-2} \int_0^1 t^{(p-1)/2}(1-t)^{(n-3)/2}dt$$

$$= \sigma_{n-1}^{-1}\sigma_{n-2}\Gamma(p/2+1/2)\Gamma(n/2-1/2)/\Gamma(n/2+p/2)$$

$$= \Gamma(n/2)\Gamma(p/2+1/2)/\Gamma(1/2)\Gamma(n/2+p/2).$$

To prove (4), we note that K^* is the set $\{\sum_{i=1}^n \varepsilon_i e_i;\ \varepsilon_i = \pm 1\}$ and contains 2 points. By (15), $v^*(\{e\}) = 2^{-n}$ for every $e \in K^*$, and by Theorem 1,

$$\mu_p^p(l_n^1) = \inf\left\{ \left(\int_{K^*} |x^*(x)|^p dv^*(x^*)\right)^{1/p}; \quad \|x\|_1 = 1\right\}$$

$$= \inf\left\{\left(\sum_{\varepsilon_i=\pm 1} 2^{-n}\left|\sum_{i=1}^n \varepsilon_i\xi_i\right|^p\right)^{1/p}; \quad \sum_{i=1}^n |\xi_i| = 1\right\}.$$

It is easily seen that $\left(\sum_{\varepsilon_i=\pm 1} 2^{-n}|\sum_{i=1}^n \varepsilon_i\xi_i|^p\right)^{1/p} = f(\{\xi_i\})$ is a convex function and $f(\{\xi_i\}) = f(\{\pm\xi_i\})$, therefore f attains its minimum on $\sum|\xi_i| = 1$ at the center of gravity of all x for which $\xi_i \geqq 0$ for all i and $\sum \xi_i = 1$, that is at the point $n^{-1}(1,1,\cdots,1)$. Hence

$$\mu_p(l_n^1) = \left(\sum_{\varepsilon_i=\pm 1} 2^{-n}n^{-p}\left|\sum_{i=1}^n \varepsilon_i\xi_i\right|^p\right)^{1/p} = \left(2^{-n}n^{-p}\sum_{k=0}^n \binom{n}{k}|n-2k|^p\right)^{1/p}.$$

In the proof of (5), we make the following observations: K — the set of extremal points of the unit cell of of G_{2n}, may be represented by the set $\{f_i = (\cos(\pi i/n), \sin(\pi i/n));\ i = 0,1,\cdots,2n-1\}$. K^* is then the set $\{g_i = \cos^{-1}(\pi/2n)(\cos((2i+1)\pi/2n), \sin((2i+1)\pi/2n)));\ i=0,1,\cdots,2n-1\}$. By (15), $v^*(\{g_i\}) = 1/2n$ for every i, and it follows from the Lemma that for every isometry T of G_{2n} and $x \in G_{2n}$,

$$\int_{K^*} |x^*(Tx)|^p dv^*(x^*) = \int_{K^*} |x^*(x)|^p dv^*(x^*).$$

Now, since $\mu_p(G_{2n}) = \min_{\|x\|=1}(\int_{K^*}|x^*(x)|^p dv^*(x^*))^{1/p}$, we conclude that the minimum is attained on the segment $[f_0, f_1]$. The points on this segment are represented by $x_t = tf_0 + (1-t)f_1$ $(0 \leqq t \leqq 1)$, and $(\int_{K^*}|x^*(x_t)|^p dv^*(x^*))^{1/p} = [(2n)^{-1}\sum_{i=0}^{2n-1}|(g_i, x_t)|^p]^{1/p}$. This is a convex function of t, and receives the same value at x_t and x_{1-t}. Hence, the minimum is attained at $x_{1/2} = \cos(\pi/2n)$ $.(\cos(\pi/2n), \sin(\pi/2n))$, and we get

$$\mu_p(G_{2n}) = \left[(2n)^{-1} \sum_{i=0}^{2n-1} |(g_i, x_{1/2})|^p\right]^{1/p} = \left[(2n)^{-1} \sum_{i=0}^{2n-1} |\cos(i\pi/n)|^p\right]^{1/p}.$$

REMARKS. The Macphail constant $\mu(X)$, is essentially the constant $\mu_1(X)$, for let $\{x_i\}_{i=1}^m$ be any finite subset of X, it is easily verified that

$$\sup_{\|x^*\| \leqq 1} \sum_{i=1}^m |x^*(x_i)| = \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^m \varepsilon_i x_i \right\| \leqq 2\sup\left(\left\| \sum_{j \in J} x_j \right\| ; J \subseteq \{1, 2, \cdots, m\}\right)$$

$$\leqq 2\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^m \varepsilon_i x_i \right\|,$$

thus $\mu_1(X) \leqq 2\mu(X) \leqq 2\mu_1(X)$. In [3, Theorem 2] it is shown that $2n\mu(X) \leqq \lambda(X)$ if X is n-dimensional. It follows from Theorem 2 above and results established in [4], that $n\mu_1(X) = \lambda(X)$ if X is any one of the spaces l_n^∞, l_n^2, l_n^1. We also get that $\lambda(G_{2n}) \geqq 2\mu_1(G_{2n})$; $\lambda(G_{2n})$ was calculated in [4].

Consider now the following generalization for μ_p. Let L be a finite dimensional real vector space and S any closed convex body with 0 in its interior. The Minkowsky functional corresponding to S is defined for each $x \in L$ by

$$\|x\|_S = \inf\{r > 0; x/r \in S\}.$$

Clearly $\|x\|_S$ is a non-negative, sublinear, positive-homogeneous function. Let $S^* = \{x \in L;\ \max_{y \in S}(x, y) \leqq 1\}$ and $\|x\|_{S^*}$ be the corresponding Minkowsky functional. It is easily verified that if $x \in L$, $\|x\|_{S^*} = \max_{y \in S}(x, y)$ and $\|x\|_S = \max_{y \in S^*}(x, y)$ (i.e. $(S^*)^* = S$). Define now $\mu_p(S)$ as $\inf \{\sup_{y \in S^*} [\sum_{i=1}^m |(x_i, y)|^p / \sum_{i=1}^m \|x_i\|_S^p]^{1/p}\}$, where the infimum ranges over all subsets $\{x_1, \cdots, x_m;\ \sum_{i=1}^m \|x_i\|_S > 0\} \subset L$. Theorem 1 and the Lemma are seen to apply, with some slight modification in the notation, for $\mu_p(S)$.

As an example, take S to be a polygon having $(2n + 1)$ equal sides in the 2-dimensional plane and 0 in its center of gravity. Exactly as in (5), $\mu_p(S) = [(2n + 1)^{-1} \sum_{i=0}^{2n} |\cos(2i\pi/(2n + 1))|^p]^{1/p}$.

5. **Proof of Theorem 3.** In the proof of (6) we use the following inequality whose proof is obvious: If X, Y are isomorphic Banach spaces, then

$$\mu_p(X) \leqq \mu_p(Y) d(X, Y), \qquad 1 \leqq p < \infty. \tag{16}$$

Using (14) and (16) we get: $1 \geqq \mu_2(X) d(X, l_n^2) \geqq \mu_2^2(X)/\mu_2(l_n^2)$, and since by (3),

$\mu_2(l_n^2) = n^{-1/2}$, we have $\mu_2(X) \leqq n^{-1/4}$. By (13), $\mu_p(X) \leqq \mu_2(X) \leqq n^{-1/4}$ if $1 \leqq p \leqq 2$, this establishes (6).

We choose now a basis $e_1, e_2, \cdots, e_n$ in X, such that $(\sum_{i=1}^n x_i^2)^{1/2} \leqq \| \sum_{i=1}^n x_i e_i \| \leqq (\sum_{i=1}^n x_i^2)^{1/2} d(X, l_n^2)$ for any set of real numbers $x_1, \cdots, x_n$. Let $e_i^* \in X^*$ be defined by $e_i^*(e_j) = \delta_{ij}$ $(1 \leqq i,j \leqq n)$. Clearly X, X^*, l_n^2 are respectively isometric to the spaces of real n-tuples $x = (x_1, x_2, \cdots, x_n)$ with the norms

$$\| x \|_X = \| \sum_{i=1}^n x_i e_i \|, \; \| x \|_{X^*} = \| \sum_{i=1}^n x_i e_i^* \|, \; \| x \|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} .$$

By the choice of the basis, we have for every x

$$\| x \|_2 d^{-1} \leqq \| x \|_{X^*} \leqq \| x \|_2 \leqq \| x \|_X \leqq \| x \|_2 d, \qquad d = d(X, l_n^2). \tag{17}$$

By Theorem 1, there are probability measures ν and ν' defined on $K = \mathrm{cl\,ext}\{x; \| x \|_X \leqq 1\}$ and $K^* = \mathrm{cl\,ext}\,\{x; \| x \|_{X^*} \leqq 1\}$ respectively, such that for every x,

$$\begin{aligned} \mu_p^p(X) \| x \|_X^p &\leqq \int_{K^*} |(x,y)|^p d\nu'(y) \\ \mu_p^p(X^*) \| x \|_{X^*}^p &\leqq \int_K |(x,v)|^p d\nu(v). \end{aligned} \tag{18}$$

Integrating the first inequality in (18) on S_{n-1} with respect to the measure $dm_{n-1}(x)$, we get by (17)

$$\mu_p^p(X) \leqq \mu_p^p(X) \int_{S_{n-1}} \| x \|_X^p dm_{n-1}(x) \leqq \int_{K^*} d\nu'(y) \int_{S_{n-1}} |(x,y)|^p \, dm_{n-1}(x).$$

In the proof of (3) we saw that

$$\int_{S_{n-1}} |(x,y)|^p dm_{n-1}(y) = \| y \|_2^p \Gamma(n/2)\Gamma(p/2 + 1/2)/\Gamma(1/2)\Gamma(n/2 + p/2),$$

which is asymptotically equivalent to $\| y \|_2^p \, n^{-p/2}$ (as a function of n), hence there is a constant a_p such that

$$\mu_p^p(X) \leqq a_p n^{-p/2} \int_{K^*} \| y \|_2^p d\nu'(y) \leqq a_p n^{-p/2} d^p, \text{ i.e. } \mu_p(X) \leqq a_p^{1/p} n^{-1/2} d,$$

and using (13) and (14) we obtain, $\mu_p^{p/2+1}(X) \leqq \mu_2(X)\mu_p(X) \leqq a_p^{1/p} n^{-1/2}$, which proves (7).

To conclude the proof, we multiply the two inequalities in (18), integrate the result on S_{n-1} with respect to $dm_{n-1}(x)$, and use the inequality $\| x \|_X \| x \|_{X^*} \geqq 1$ (if $x \in S_{n-1}$). We obtain

$$(19) \qquad \mu_p^p(X)\mu_p^p(X^*) \leqq \int_{K^*} dv'(y) \int_K dv(v) \int_{S_{n-1}} |(x,y)(x,v)|^p dm_{n-1}(x).$$

We first estimate $J = \int_{S_{n-1}} |(x,y)(x,v)|^p dm_{n-1}(x)$ for the special case $y = (1,0,\cdots,0)$, $v = (\cos\theta, \sin\theta, 0, \cdots, 0)$. Let $0 \leqq \theta_1 \leqq \pi$, we may represent the points $x = (x_1, \cdots, x_n) \in S_{n-1}$ by

$$x_1 = \cos\theta_1 \sin\theta_2,\ x_2 = \sin\theta_1 \sin\theta_2,\ x_3 = \cos\theta_2 \sin\theta_3,$$
$$x_4 = \cos\theta_2 \cos\theta_3 \sin\theta_4, \cdots, x_{n-1} = \cos\theta_2 \cdots \cos\theta_{n-2} \sin\theta_{n-1},$$
$$x_n = \cos\theta_2 \cdots \cos\theta_{n-1}, \text{ where } -\pi \leqq \theta_{n-1} \leqq \pi,\ -\pi/2 \leqq \theta_i \leqq \pi/2$$

$(2 \leqq i \leqq n-2)$, and $dm_{n-1}(x) = (\pi\sigma_{n-2})^{-1} \cos^{n-3}\theta_2 \cdots \cos\theta_{n-2} d\theta_1 \cdots d\theta_{n-1}$. Then

$$\begin{aligned} J &= \int_{S_{n-1}} |\sin^2\theta_2 \cos\theta_1 \cos(\theta_1 - \theta)|^p dm_{n-1}(x) \\ &= \left(\pi^{-1}\int_0^\pi |\cos\theta_1 \cos(\theta_1 - \theta)|^p d\theta_1\right)\left(2\sigma_{n-2}^{-1}\sigma_{n-3} \int_0^{\pi/2} \sin^{2p}\theta_2 \cos^{n-3}\theta_2 d\theta_2\right) \\ &\leqq 2\sigma_{n-2}^{-1}\sigma_{n-3} \int_0^{\pi/2} \sin^{2p}\theta_2 \cos^{n-3}\theta_2 d\theta_2 = (\text{substituting, } t = \sin^2\theta_2) \\ &= \sigma_{n-2}^{-1}\sigma_{n-3} \int_0^1 t^{p-1/2} (1-t)^{n/2-2} dt \\ &= \sigma_{n-2}^{-1}\sigma_{n-3}\Gamma(p+1/2)\Gamma(n/2-1)/\Gamma(p+n/2-1/2), \end{aligned}$$

which is asymptotically equivalent to n^{-p}. Substituting this estimate of J in (19) we get

$$\mu_p^p(X)\mu_p^p(X^*) \leqq b_p n^{-p} \int_{K^*} \| y \|_2^p dv'(y) \int_K \| v \|_2^p dv(v) \leqq \text{(By (17))}$$

$$(20) \qquad \leqq b_p n^{-p} d^p \int_{K^*} 1^p dv'(y) \int_K 1^p dv(v) = b_p n^{-p} d^p.$$

Using in (20) the well known inequality, $d(X, l_n^2) \leqq \sqrt{n}$ [5], which holds for every real n-dimensional Banach space X, we obtain (10).

Rewriting (20) as $\mu_p(X)\mu_p(X^*) \leqq a_p n^{-1} d$, multiplying by $\mu_2(X)$ and using (14), we get $\mu_2(X)\,\mu_p(X)\mu_p(X^*) \leqq a_p n^{-1}$, and by (13) $\mu_p^{1+p/2}(X)\mu_p(X^*) \leqq a_p n^{-1}$ if $p \geqq 2$. Similarly $\mu_p^{1+p/2}(X^*)\mu_p(X) \leqq a_p n^{-1}$, and (9) follows by multlplying the last two inequalities. Since $\mu_p(X)\mu_p(X^*) \leqq \mu_2(X)\,\mu_2(X^*) \leqq c_2 n^{-2/3}$ if $1 \leqq p \leqq 2$, (8) is also established.

REMARK. A result of [2, Theorem 4] is that $\mu_1(X) \leqq 2n^{-1/4}$; (6) strengthens this inequality, and an even sharper one for $n \geqq 3$ and $p = 1$ is $\mu_1(X) \leqq \mu_1^{1/2}(l_n^2) = (\Gamma(n/2)\Gamma(1/2)/\Gamma(n/2+1/2))^{1/2} \leqq (2/\pi(n-1))^{1/4}$. We do not know whether the upper bounds of (6) and (7) are the best possible, and we think that they can be improved to $c_p n^{-1/2}$ and $c_p n^{-1/p}$ respectively.

6. **Proof of Theorem 4.** We proved in [3] that $K_G\mu(X)d(X,l_n^2)d(X,l_n^1) \geqq 1$ and it is easily seen that the proof given there actually yields (11).

The proof of (12) is a direct application of Theorem 4.3 [6]. To see this, we reproduce Definition 3.1 [6]: A Banach space X is called an $\mathscr{L}_{p,\lambda}$ space, $1 \leqq p \leqq \infty$, $1 \leqq \lambda < \infty$, if for every finite dimensional subspace B of X, there is a finite dimensional subspace E of X containing B, such that $d(E,l_n^p) \leqq \lambda$, $(n = \dim E)$.

A result arrived at in Theorem 4.3 [6] is: Let X be an $\mathscr{L}_{\infty,\lambda}$ space, and let Y be an $\mathscr{L}_{p,\rho}$ space, $1 \leqq p \leqq 2$. For any finite subset $\{x_i\}_{i=1}^m \subset X$, and any linear bounded operator T from X into Y, the following inequality holds:

$$\left(\sum_{i=1}^m \| Tx_i \|^2\right)^{1/2} \leqq K_G\lambda\rho \| T \| \sup_{\|x^*\| \leqq 1} \left(\sum_{i=1}^m |x^*(x_i)|^2\right)^{1/2}. \tag{21}$$

Since every finite dimensional space X is the limit of a sequence of polyhedral spaces, and since the constants which appear in (12) are continuous functions of X, we may assume X to be a polyhedral space. Then, embedding X in a suitable l_N^∞ space, let $P: l_N^\infty \to X$ be a projection on X such that $\lambda(X) = \| P \|$; there is such P, for by [1], $\lambda(X) = \min\{\| P \|; P \text{ is a projection of } l_N^\infty \text{ onto } X\}$. Let $T: X \to l_n^p$ be an isomorphism such that $\| T \|\ \| T^{-1} \| = d(X,l_n^p)$. Then $TP: l_N^\infty \to l_n^p$, and by (21) we have for any subset $\{x_i\}_1^m \subset l_N^\infty$

$$\left(\sum_{i=1}^m \| TPx_i \|^2\right)^{1/2} \leqq K_G \| TP \| \sup\left\{\left(\sum_{i=1}^m |y^*(x_i)|^2\right)^{1/2};\ y^* \in l_N^1, \| y^* \| \leqq 1\right\}. \tag{22}$$

Taking in particular a subset $\{x_i\}_1^m \subset X$, and noting that $\sup\{(\sum_{i=1}^m |y^*(x_i)|^2)^{1/2};$ $y^* \in l_N^1, \| y^* \| \leqq 1\} = \sup\{(\sum_{i=1}^m |x^*(x_i)|^2)^{1/2};\ x^* \in X^*, \| x^* \| \leqq 1\}$, we get from (22)

$$\left(\sum_{i=1}^{m}\|x_i\|^2\right)^{1/2} \leqq \|T^{-1}\|\left(\sum_{i=1}^{m}\|Tx_i\|^2\right)^{1/2}$$

$$\leqq K_G\|T^{-1}\|\,\|T\|\,\|P\|\sup\{\left(\sum_{i=1}^{m}|x^*(x_i)|^2\right)^{1/2} ;\ x^*\in X^*,\ \|x^*\|\leqq 1\}$$

$$= K_G d(X,l_n^p)\lambda(X)\sup\{\left(\sum_{i=1}^{m}|x^*(x_i)|^2\right)^{1/2} ;\ x^*\in X^*,\ \|x^*\|\leqq 1\},$$

which yields (12).

Using (21) we can improve on Corollary 2 [3],

COROLLARY. *Let X be an n-dimensional subspace of l^1, then*

$$\lambda(X)\geqq K_G^{-2/3}\, n^{1/3}.$$

Proof. Let $I: X\to l^1$ be tbe formal identity operator on X, and J be the operator embedding X in l^∞ and let $\varepsilon>0$. $Y=JX$ is isometric to X, hence there is a projection P of l^∞ onto Y such that $\|P\|\leqq\lambda(X)+\varepsilon$. Now, $IJ^{-1}P$ maps the $\mathscr{L}_{\infty,1+\varepsilon}$ space l^∞ into the $\mathscr{L}_{1,1+\varepsilon}$ space l^1, so that for any subset $\{y_i\}_1^m\subset l^\infty$ we have by (21),

$$\left(\sum_{i=1}^{m}\|IJ^{-1}Py_i\|^2\right)^{1/2} \leqq K_G(1+\varepsilon)^2\|IJ^{-1}P\|\sup\left\{\left(\sum_{i=1}^{m}|y^*(y_i)|^2\right)^{1/2};\ \|y^*\|\leqq 1\right\}.$$

If in particular $\{y_i\}_1^m\subset Y$, we get $\|IJ^{-1}Py_i\| = \|y_i\|$ and

$$\sup\left\{\sum_{i=1}^{m}|y^*(y_i)|^2;\ \|y^*\|\leqq 1, y^*\in(l^\infty)^*\right\} = \sup\left\{\sum_{i=1}^{m}|y^*(y_i)|^2;\|y^*\|\leqq 1, y^*\in Y^*\right\},$$

from which it follows that $1\leqq K_G(1+\varepsilon)^2(\lambda(X)+\varepsilon)\mu_2(JX)$, and since ε was arbitrary and JX isometric to X, we obtain $K_G\mu_2(X)\lambda(X)\geqq 1$.

Applying now the inequality $\mu_2^2(X)\leqq\mu_1(X)\leqq 2\mu(X)\leqq n^{-1}\lambda(X)$, the proof is concluded.

7. **Proof of Theorem 5.** We shall write $f(n)\lesssim g(n)$ (or $g(n)\gtrsim f(n)$) whenever f, g are positive functions defined on the integers and $\sup_n(f(n)/g(n))<\infty$. It is easily verified from (3) that $\mu_p(l_n^2)\sim n^{-1/2}$ for fixed values of p, and since $d(l_n^2,l_n^r) = n^{|r/2-1/u|}$ for any $1\leqq r\leqq\infty$ (see [3] for information and reference), we get by (16)

$$\mu_p(l_n^r)\leqq\mu_p(l_n^2)d(l_n^2,l_n^r)\sim n^{-1/r},\quad \text{if } 2\leqq r\leqq\infty. \tag{23}$$

Let $r'=r/(r-1)$ and $\{e_i\}_{i=1}^n$, $\{e_i^*\}_{i=1}^n$ be the natural bases of l_n^r and $l_n^{r'}=(l_n^{r*})$

respectively, and let $\theta(\{e_i^*\}) = 1/n$ for every $1 \leqq i \leqq n$. It follows from the second part of Theorem 1, that

$$(24)\qquad \mu_p(l_n^r) \geqq \inf_{\|x\|_r=1} \left(\sum_{i=1}^n |e_i^*(x)|^p n^{-1}\right)^{1/p} = n^{-1/p} \inf(\|x\|_p/\|x\|_r) =$$
$$= \min\{n^{-1/p}, n^{-1/r}\}.$$

Combining (23) and (24), we get

$$(25)\qquad \mu_p(l_n^r) \sim n^{-1/r}, \quad \text{if} \quad p \geqq r \geqq 2.$$

By the definition of $\mu_p(l_n^r)$ we have that

$$(26)\qquad \mu_p(l_n^r) \leqq \sup_{\|x^*\|=1} \left(\sum_{i=1}^n |x^*(e_i)|^p \Big/ \sum_{i=1}^n \|e_i\|^p\right)^{1/p}$$
$$= n^{-1/p} \sup(\|x\|_p/\|x\|_{r'}) = \max\{n^{-1/p}, n^{-1/r'}\},$$

and on taking (24) and (26) we see that

$$(27)\qquad \mu_p(l_n^r) \sim n^{-1/p}, \text{ if } r \geqq p \geqq r'.$$

Using the equivalence $\mu_1(X) \leqq 2\mu(X) \leqq 2\mu_1(X)$ and the result

$$\mu(l_n^r) \sim \min\{n^{-1/2}, n^{-1/r'}\}$$

of [3] and (13), we obtain

$$(28)\qquad \mu_p(l_n^r) \geqq \mu_1(l_n^r) \sim \min\{n^{-1/2}, n^{-1/r'}\}, \text{ for } 1 \leqq r \leqq \infty.$$

If $r \geqq r' \geqq p \geqq 1$ we get by (28) $\mu_p(l_n^r) \gtrsim n^{-1/r'}$, and by (26) $\mu_p(l_n^r) \leqq n^{-1/r'}$ that is

$$(29)\qquad \mu_p(l_n^r) \sim n^{-1/r'}, \text{ if } r \geqq r' \geqq p \geqq 1.$$

Summing up, we see that equations (25), (27) and (29) conclude with the cas $r \geqq 2$, $\infty > p \geqq 1$.

When $1 < r \leqq 2$, it follows from (20) that

$$\mu_p(l_n^r)\, \mu_p(l_n^{r'}) \lesssim n^{-1} d(l_n^2, l_n^r) = n^{|1/2-1/r|-1},$$

whence by (25), $\mu_p(l_n^r) \lesssim n^{-1/2}$ if $p \geqq r' \geqq 2$. For $p = 1$ $\mu_1(l_n^r) \sim \mu(l_n^r) \sim n^{-1/2}$[3], so that on using (13) we have for $p \geqq r' \geqq 2$ and $r' \geqq q \geqq 1$,

$$n^{-1/2} \lesssim \mu_1(l_n^r) \leqq \mu_q(l_n^r) \leqq \mu_p(l_n^r) \lesssim n^{-1/2},$$

that is $\mu_p(l_n^r) \sim n^{-1/2}$ for every $1 < r \leqq 2$ and $1 \leqq p < \infty$.

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